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Coloring of distance graphs with intervals as distance sets[☆]

Peter Che Bor Lam[a], Wensong Lin[a,b]

[a] *Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong*
[b] *Department of Applied Mathematics, Southeast University, Nanjing 210096, PR China*

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Abstract

Let D be a set of positive integers. The distance graph $G(Z, D)$ with distance set D is the graph with vertex set Z in which two vertices x, y are adjacent if and only if $|x - y| \in D$. The fractional chromatic number, the chromatic number, and the circular chromatic number of $G(Z, D)$ for various D have been extensively studied recently. In this paper, we investigate the fractional chromatic number, the chromatic number, and the circular chromatic number of the distance graphs with the distance sets of the form $D_{m,[k,k']} = \{1, 2, \ldots, m\} - \{k, k+1, \ldots, k'\}$, where m, k, and k' are natural numbers with $m \geq k' \geq k$. In particular, we completely determine the chromatic number of $G(Z, D_{m,[2,k']})$ for arbitrary m, and k'.
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1. Introduction

The fractional chromatic number of a graph is a well-known variation of the chromatic number. Let G be a graph. A *fractional coloring* of G is a mapping c from $\Gamma(G)$, the set

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E-mail address: cblam@math.hkbu.edu.hk (P.C.B. Lam).

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of all independent sets of G, to the interval $[0, 1]$ such that $\sum_{x \in I \in \Gamma(G)} c(I) \geq 1$ for all vertices $x \in G$. The *weight* of c is $w(c) = \sum_{I \in \Gamma(G)} c(I)$. The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is: $\inf\{w(c) \mid c$ is a fractional coloring of $G\}$.

The circular chromatic number of a graph is a natural generalization of the chromatic number. It was introduced by Vince [14] as the star chromatic number of a graph, see also [1]. Given two positive integers p and q with $p \geq 2q$, a (p, q)-coloring of the graph $G = (V, E)$ is a mapping ϕ from V to $\{0, 1, \ldots, p-1\}$, such that $\|\phi(x) - \phi(y)\|_p \geq q$ for any edge $xy \in E$, where $\|a\|_p = \min\{|np + a|, |np - a| : n \in Z\}$. The *circular chromatic number* of G, denoted by $\chi_c(G)$ of G is: $\inf\{p/q \mid$ there exists a (p, q)-coloring of $G\}$.

A $(p, 1)$-coloring of a graph G is simply an ordinary p-coloring of G. It was proved in [22] that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ for any graph G. In fact, $\chi_c(G)$ is a refinement of $\chi(G)$ containing more information on the structure of G. It is usually more difficult to determine the circular chromatic number of a graph than its chromatic number. For any graph G, it is well-known that

$$\max\left\{\omega(G), \frac{|V(G)|}{\alpha(G)}\right\} \leq \chi_f(G) \leq \chi_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G), \tag{1}$$

where $\omega(G)$ and $\alpha(G)$ are the clique number and the independence number of G respectively.

Given a set D of positive integers, the *distance graph* $G(Z, D)$ has vertex set Z, the set of all integers Z. Two vertices j and j' in Z are adjacent if and only if $|j - j'| \in D$. We call D the *distance set*. The fractional chromatic number, the circular chromatic number, the chromatic number and the clique number of a distance graph $G(Z, D)$ are denoted by $\chi_f(Z, D)$, $\chi_c(Z, D)$, $\chi(Z, D)$ and $\omega(Z, D)$, respectively.

The problem of determining $\chi(Z, D)$ for various types of distance sets D has been studied extensively [3–8,10,12,13,15,16]. Recently $\chi_c(Z, D)$ for several types of distance sets D has also been investigated [2,3,9,17,19–21].

Throughout this paper, m, k, and s will be natural numbers; and unless otherwise stated, β will be an integer with $0 \leq \beta \leq k - 1$. Given any integers a and b, $[a, b]$, $k[a, b]$ and $D_{m,k,s}$ shall denote the sets $\{x : x \in Z, a \leq x \leq b\}$, $\{kx : x \in Z, a \leq x \leq b\}$ and $[1, m] \setminus k[1, s]$ respectively. By combining several results in [9,13,20], we have the following three theorems.

Theorem 1.1. *If* $m < (s + 1)k$, *then*

$$\omega(Z, D_{m,k,s}) = \chi_f(Z, D_{m,k,s}) = \chi_c(Z, D_{m,k,s}) = k.$$

Theorem 1.2. *If* $m \geq (s + 1)k$, *then* $\chi_f(Z, D_{m,k,s}) = \frac{m+sk+1}{s+1}$.

Theorem 1.3. *Suppose* $m \geq (s + 1)k$ *and* $d = \gcd(k, m + sk + 1)$. *If* $d \neq 1$ *and* $d(s+1) \nmid (m+sk+1)$, *then* $\chi_c(Z, D_{m,k,s}) = \frac{m+sk+2}{s+1}$; *otherwise* $\chi_c(Z, D_{m,k,s}) = \frac{m+sk+1}{s+1}$.

Using Theorems 1.1–1.3 and (1), one can easily obtain $\chi(Z, D_{m,k,s})$. Given non-negative integer β, let $D_{m,[k,k+\beta]}$ denote the set $[1, m] \setminus [k, k + \beta]$. Theorems 1.4–1.7 can be found in [17].

Theorem 1.4. *If $m < 2k$, then*

$$\omega(Z, D_{m,[k,k+\beta]}) = \chi_f(Z, D_{m,[k,k+\beta]}) = \chi_c(Z, D_{m,[k,k+\beta]}) = k.$$

Theorem 1.5. *If $2k \le m < 2k + 2\beta$ and $1 \le \beta \le k - 1$, then*

$$\chi_f(Z, D_{m,[k,k+\beta]}) = \chi_c(Z, D_{m,[k,k+\beta]}) = \frac{m+1}{2}.$$

Theorem 1.6. *If $m \ge 2k + 2\beta$ and $1 \le \beta \le k - 1$, then*

$$\chi_f(Z, D_{m,[k,k+\beta]}) = \frac{m+k+1}{2}.$$

Theorem 1.7. *Suppose $m \ge 2k + 2\beta$ and $1 \le \beta \le k - 1$. Let $m + k + 1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers; m' and k' are odd integers.*

(a) *If $1 \le r \le s$, then $\frac{m+k+1}{2} + \frac{1}{3} \le \chi_c(Z, D_{m,[k,k+\beta]}) \le \frac{m+k+2}{2}$;*

(b) *If $r = 0$ and* $\gcd(m + k + 1, k) \ne 1$, *then $\frac{m+k}{2} + \frac{2}{3} \le \chi_c(Z, D_{m,[k,k+\beta]}) \le \frac{m+k+2}{2}$;*

(c) *Otherwise, $\chi_c(Z, D_{m,[k,k+\beta]}) = \frac{m+k+1}{2}$.*

Also, using Theorems 1.4–1.7 and (1), one can obtain $\chi(Z, D_{m,[k,k+\beta]})$ for $1 \le \beta \le k - 1$. Recently, the result in Theorem 1.7(b) has been improved to $\chi_c(Z, D_{m,[k,k+\beta]}) = \frac{m+k+2}{2}$,—[11].

Theorem 1.8 ([2]). *If $D = [1, m] \cup \{n\}$, where $1 \le m < n$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = \begin{cases} m+1, & \text{if } n \not\equiv 0 \pmod{m+1}, \\ m+1+1/k, & \text{if } n = k(m+1). \end{cases}$$

Theorem 1.9 ([2]). *If $D = [q, p]$, where $q \le p$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = 1 + p/q.$$

For $0 \le \beta \le k - 1$, let $D_{m,[k,sk+\beta]}$ denote the set $[1, m] \setminus [k, sk + \beta]$, where m, k, and s are natural numbers. In this paper, we consider various chromatic numbers of the distance graph $G(Z, D_{m,[k,sk+\beta]})$. If $s = 1$ then $G(Z, D_{m,[k,sk+\beta]})$ is simply $G(Z, D_{m,[k,k+\beta]})$. Theorems 1.4–1.7 are results for this case. Results for the case $k = 1$ is given in Theorem 1.9. So we assume $s \ge 2$ and $k \ge 2$ throughout the paper.

In Section 2, we first determine χ_f, χ, and χ_c of $G(Z, D_{m,[k,sk+\beta]})$ for $m \le (s+1)k + \beta - 1$, where $1 \le \beta \le k - 1$. And then we determine $\chi_f(Z, D_{m,[k,sk+\beta]})$ and $\chi(Z, D_{m,[k,sk+\beta]})$ for $m \ge (s+2)k+\beta-1$ with $0 \le \beta \le k-1$. However, for $m \ge (s+2)k+\beta-1$, exact values of $\chi_c(Z, D_{m,[k,sk+\beta]})$ are determined only under certain conditions, whereas in general, we have given an upper bound and a lower bound of $\chi_c(Z, D_{m,[k,sk+\beta]})$. In Section 3, we investigate the fractional chromatic number and chromatic number of $G(Z, D_{m,[k,sk+\beta]})$ when $(s+1)k+\beta \le m \le (s+2)k+\beta-2$. In particular, we completely determine the chromatic number of $G(Z, D_{m,[2,k']})$ for arbitrary integers m and k'.

2. $m \le (s+1)k+\beta-1$ or $m \ge (s+2)k+\beta-1$

Because $G(Z, D_{m,[k,sk+\beta]})$ is a subgraph of $G(Z, D_{m,k,s})$ and $G(Z, D_{m,[k,sk+\beta]})$ has a clique of order k, Theorem 2.1 follows from Theorem 1.1.

Theorem 2.1. *If $m < (s+1)k$, then*

$$\omega(Z, D_{m,[k,sk+\beta]}) = \chi_f(Z, D_{m,[k,sk+\beta]}) = \chi_c(Z, D_{m,[k,sk+\beta]}) = \chi(Z, D_{m,[k,sk+\beta]}) = k.$$

Lemma 2.2 ([2]). *Suppose D is a set of positive integers, and that p and q are positive integers. Let $d_D(p,q) = \min\{\|qj\|_p : j \in D\}$. If $d_D(p,q) \ge 1$, then $\chi_c(Z, D) \le p/d_D(p,q)$.*

For $x \le y$, denote by $[x, y]$ and $G[x, y]$ the vertex set $\{x, x+1, \dots, y\}$ and the subgraph it induces in $G(Z, D_{m,[k,sk+\beta]})$ respectively.

Theorem 2.3. *Suppose $1 \le \beta \le k-1$. If $(s+1)k \le m < (s+1)k+\beta$, then $\chi_f(Z, D_{m,[k,sk+\beta]}) = \chi_c(Z, D_{m,[k,sk+\beta]}) = (m+1)/(s+1)$.*

Proof. We can prove that $\alpha(G[0, m]) = s+1$. So $\chi_c(Z, D_{m,[k,sk+\beta]}) \ge \chi_f(Z, D_{m,[k,sk+\beta]}) \ge (m+1)/(s+1)$ follows from (1). It remains to show that $\chi_c(Z, D_{m,[k,sk+\beta]}) \le (m+1)/(s+1)$. Consider $d_{D_{m,[k,sk+\beta]}}(m+1, s+1) = \min\{\|(s+1)j\|_{m+1} : j \in [1, k-1] \cup [sk+\beta+1, m]\}$.

If $1 \le j \le k-1$ then $s+1 \le (s+1)j \le (s+1)(k-1) \le m+1-(s+1)$ and therefore $\|(s+1)j\|_{m+1} \ge s+1$.

If $sk+\beta+1 \le j \le m$ then $(s+1)j \le (s+1)m = (s+1)(m+1)-(s+1)$, $(s+1)j \ge (s+1)(sk+\beta+1) \ge (s+1)(m-k+2) = s(m+1)+(s+1)+m+1-(s+1)k \ge s(m+1)+(s+1)$; and therefore $\|(s+1)j\|_{m+1} \ge s+1$.

It follows that $d_{D_{m,[k,sk+\beta]}}(m+1, s+1) \ge s+1$ and by Lemma 2.2, $\chi_c(Z, D_{m,[k,sk+\beta]}) \le (m+1)/(s+1)$. □

From Theorems 2.1 and 2.3, we obtain the following corollary, which is equivalent to Theorem 1.8.

Corollary 2.4. *If $D = [k, m]$, where $1 < k \le m$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = \begin{cases} k, & \text{if } m \not\equiv 0 \pmod{k}, \\ k + 1/(s+1), & \text{if } m = (s+1)k. \end{cases}$$

For the rest of this section we assume that $m \ge (s+1)k+\beta$.

Lemma 2.5. *If $s \ge 2$, then $\alpha(G[0, m+sk]) = s+1$ if and only if $m \ge (s+2)k+\beta-1$.*

Proof. Suppose $m \ge (s+2)k+\beta-1$ and S is a maximum independent set of $G[0, m+sk]$. Without loss of generality, we can assume $0 \in S$ and so $S \subseteq [0, sk+\beta] \cup [m+1, m+sk]$. Let $X = S \cap [0, sk+\beta] = \{x_1, x_2, \dots, x_a\}$ and $Y = S \cap [m+1, m+sk] = \{y_1, y_2, \dots, y_b\}$, where $0 = x_1 < x_2 < \cdots < x_a \le sk+\beta < m+1 \le y_1 < y_2 < \cdots < y_b \le m+sk$. Then

$$x_{i+1} - x_i \ge k \qquad \text{and} \qquad y_{j+1} - y_j \ge k \tag{2}$$

for $i \in [1, a-1]$ and $j \in [1, b-1]$. Moreover, for all $i \in [1, a-1]$, put $x_{i+1} - x_i = r_i k + \gamma_i$ and $Z_i = \{z : x_i + m + 1 \le z \le x_{i+1} + (sk + \beta)\}$, where $r_i \ge 1$ and $0 \le \gamma_i \le k - 1$. Then

$$k(|S \cap Z_i| - 1) + 1 \le |Z_i| = x_{i+1} + (sk + \beta) - [x_i + m + 1] + 1 \le (r_i - 1)k. \tag{3}$$

Since the last term of Eq. (3) is a multiple of k, we have

$$|S \cap Z_i| \le r_i - 1, \tag{4}$$

and

$$x_a = \sum_{i=1}^{a-1}(x_{i+1} - x_i) \ge k(a-1) + k\sum_{i=1}^{a-1}(r_i - 1) \ge k(a-1) + k\sum_{i=1}^{a-1}|S \cap Z_i|. \tag{5}$$

Now put $Z_a = \{z : x_a + m + 1 \le z \le m + sk\}$. Then

$$sk - x_a = |Z_a| \ge (|S \cap Z_a| - 1)k + 1. \tag{6}$$

Note that (6) holds even if $sk \le x_a$, in which case $Z_a = S \cap Z_a = \varnothing$. Because $Y = \bigcup_{i=1}^{a} S \cap Z_i$, we have $(s+2)k \ge (a+b)k + 1$, and so $a + b = |S| \le s + 1$.

Since $\{ik : i = 0, 1, \ldots, s\}$ is an independent set of $G[0, m + sk]$ with $s + 1$ vertices, therefore $|S| \ge s + 1$, and consequently $\alpha(G[0, m + sk]) = s + 1$.

If $m < (s+2)k + \beta - 1$, then $\{0, 2k - 1, 3k - 1, \ldots, sk - 1, m + 1, m + sk\}$ is an independent set of $G[0, m + sk]$ with $s + 2$ vertices and therefore $\alpha(G[0, m + sk]) \ge s + 2$. So $\alpha(G[0, m + sk]) = s + 1$ if and only if $m \ge (s+2)k + \beta - 1$. $\square$

Suppose $m \ge (s+2)k + \beta - 1$ and $h \le s$. If S' is an independent set of $G[0, m + sk]$ such that $S' \subseteq [0, hk + \beta] \cup [m + 1, m + hk]$, then $|S'| \le h + 1$ as in Lemma 2.5. In fact, we obtain the following lemma.

Corollary 2.6. *Suppose $m \ge (s+2)k + \beta - 1$ and $h \le s$. If S' is an independent set of $G[u, u + m + sk]$ such that $u \in S' \subseteq [u, u + (h+1)k - 1] \cup [u + m + 1, u + m + hk]$, then $|S'| \le h + 1$.*

Since $G(Z, D_{m,[k,sk+\beta]})$ is a subgraph of $G(Z, D_{m,k,s})$, the following two theorems follow from Theorems 1.2 and 1.3, Lemma 2.5 and (1). For the rest of this paper, we put $m' = m + sk + 1$ and $d = \gcd(m', k) = \gcd(m + 1, k)$.

Theorem 2.7. *Suppose $s \ge 2$. If $m \ge (s+2)k + \beta - 1$ then $\chi_f(Z, D_{m,[k,sk+\beta]}) = m'/(s+1)$.*

Theorem 2.8. *Suppose $s \ge 2$ and $m \ge (s+2)k + \beta - 1$. If $d = 1$ or $d(s+1) \mid m'$ then $\chi_c(Z, D_{m,[k,sk+\beta]}) = m'/(s+1)$; otherwise,*

$$m'/(s+1) \le \chi_c(Z, D_{m,[k,sk+\beta]}) \le (m'+1)/(s+1).$$

Except for the case $(s+1) \mid m'$ but $d(s+1) \nmid m'$, the bounds of $\chi_c(Z, D_{m,[k,sk+\beta]})$ given by Theorem 2.8 are sharp enough to determine $\chi(Z, D_{m,[k,sk+\beta]})$ when $s \ge 2$ and $m \ge (s+2)k + \beta - 1$. Theorem 2.11 will deal with the exceptional case. We shall first prove the following two lemmas.

Lemma 2.9. *Suppose $m \geq (s+2)k+\beta$ and S is an independent set of $G[u, u+m'-k]$ with $s+1$ vertices. If $u \in S \subseteq [u, u+sk] \cup [u+m+1, u+m'-k]$, then*

$$S = \{u, u+k, \ldots, u+lk, u+lk+m+1, u+(l+1)k+m+1, \ldots, u + (s-1)k+m+1\}$$

for some integer $l \in [0, s]$, where $u+(s-1)k+m+1 = u+m'-k$. Consequently, if $u+m'-k \notin S$, then $S = \{u, u+k, \ldots, u+sk\}$.

Proof. We may assume $u = 0$ and so $A \subseteq [0, sk] \cup [m+1, m'-k]$. Define $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$, where $0 = x_1 < x_2 < \cdots < x_a \leq sk < m+1 \leq y_1 < y_2 < \cdots < y_b \leq m'-k$. Then

$$x_{i+1} - x_i \geq k \qquad \text{and} \qquad y_{j+1} - y_j \geq k \tag{7}$$

for $i \in [1, a-1]$ and $j \in [1, b-1]$. We also put $Z_i = \{z : x_i + m + 1 \leq z \leq x_{i+1} + (sk+\beta)\}$ for all $i \in [1, a-1]$. Then we have

$$\begin{aligned} &x_{i+1} + (sk+\beta) - [x_i + m + 1] + 1 \\ &\quad = x_{i+1} - x_i - [m - (sk - \beta)] \leq x_{i+1} - x_i - 2k. \end{aligned} \tag{8}$$

So if $i \in [1, a-1]$ and $S \cap Z_i \neq \emptyset$, then $x_{i+1} - x_i > 2k$ and

$$x_{i+1} - x_i - 2k \geq (|S \cap Z_i| - 1)k + 1. \tag{9}$$

Denote by I^* the subset of $[1, a-1]$ such that $S \cap Z_i \neq \emptyset$ if $i \in I^*$. Then

$$x_a = \sum_{i=1}^{a-1} (x_{i+1} - x_i) \geq k(a-1) + \sum_{i \in I^*} (k|S \cap Z_i| + 1). \tag{10}$$

Now put $Z_a = \{z : x_a + m + 1 \leq z \leq m' - k\}$. We can prove that $Y = \bigcup_{i=1}^{a} (S \cap Z_i)$. Moreover, if $(s-1)k > x_a - 1$, then

$$(s-1)k - x_a + 1 \geq (|S \cap Z_a| - 1)k + 1. \tag{11}$$

Note that (11) holds even if $(s-1)k \leq x_a - 1$, in which case $Z_a = \emptyset$. Summing up (10) and (11), we have

$$(s-1)k \geq (a+b-2)k + |I^*| = (s-1)k + |I^*|, \tag{12}$$

which is possible only if $I^* = \emptyset$, and equality signs in (10)–(12) hold. So $x_a = (a-1)k$. If $S \cap Z_a \neq \emptyset$, then $|Z_a| = (s-1)k - x_a + 1 = (s-a)k + 1$ and $|S \cap Z_a| = s - a + 1$. It follows that $Y = S \cap Z_a = \{lk+m+1, (l+1)k+m+1, \ldots, (s-1)k+m+1 = m'-k\}$ and $X = \{0, k, \ldots, lk\}$, where $l = a-1$. If $S \cap Z_a = \emptyset$, which is necessarily the case if $m' - k \notin S$, then $S = X = \{0, k, \ldots, sk\}$. $\square$

Lemma 2.10. *Suppose $m = (s+2)k + \beta - 1$ and A is an independent set of $G[u, u+m'-k]$ with $s+1$ vertices. If $u \in A \subseteq [u, u+sk] \cup [u+m+1, u+m'-k]$ then*

$$A \subseteq \{u, u+k, \ldots, u+sk, u+m+1, u+k+m+1, \ldots, u+(s-1)k + m + 1\}.$$

Proof. Without loss of generality, we can assume $u = 0$ and so $A \subseteq [0, sk] \cup [m+1, m'-k]$. Let $X = S \cap [0, sk] = \{x_1, x_2, \ldots, x_a\}$ and $Y = S \cap [m+1, m'-k] = \{y_1, y_2, \ldots, y_b\}$, where $0 = x_1 < x_2 < \cdots < x_a \le s_k + \beta < m+1 \le y_1 < y_2 < \cdots < y_b \le m' - k$. Then

$$x_{i+1} - x_i \ge k \qquad \text{and} \qquad y_{j+1} - y_j \ge k \tag{13}$$

for $i \in [1, a-1]$ and $j \in [1, b-1]$. Put $Z_i = \{z : x_i + m + 1 \le z \le x_{i+1} + (sk + \beta)\}$ for all $i \in [1, a-1]$. If $S \cap Z_i \ne \varnothing$, then

$$(|S \cap Z_i| - 1)k \le x_{i+1} + (sk + \beta) - (x_i + m + 1)$$

and so

$$(|S \cap Z_i| + 1)k \le x_{i+1} - x_i. \tag{14}$$

Now put $Z_a = \{z : x_a + m + 1 \le z \le m' - k\}$. If $S \cap Z_a \ne \varnothing$, then

$$(|S \cap Z_a| - 1)k \le (m' - k) - (x_a + m + 1)$$

and so

$$(|S \cap Z_a| + 1)k \le (s + 1)k - x_a. \tag{15}$$

Note that (14) and (15) hold even if $S \cap Z_i = \varnothing$. If any one of the inequality signs in (14) or (15) is strict, then upon summing up all these inequalities, we get

$$(a + b)k < (s + 1)k,$$

which is a contradiction. So all inequality signs in (14) or (15) are in fact equal signs. It follows that for all $i \in [1, a]$, $x_i = l_i k$ for some non-negative integer l_i. Moreover, if $S \cap Z_i \ne \varnothing$, then $|S \cap Z_i| = l_{i+1} - l_i - 1$, and each element in $S \cap Z_i$ must be of the form $x_i + l_{ij}k + m + 1 = (l_i + l_{ij})k + m + 1$ for some non-negative integer l_{ij}. We can also verify that $l_i + l_{ij} \le s + 1$. □

Now we are ready to obtain $\chi(Z, D_{m,[k,sk+\beta]})$ when $m \ge (s+2)k + \beta - 1$, $(s+1) \mid m'$ and $d(s+1) \nmid m'$.

Theorem 2.11. *Suppose $s \ge 2$ and $m \ge (s+2)k + \beta - 1$. If $d \ne 1$ and $d(s+1) \nmid m'$ then $\chi(Z, D_{m,[k,sk+\beta]}) = \lceil (m'+1)/(s+1) \rceil$.*

Proof. Consider the subgraph $G[0, 2m'-1]$. By Lemma 2.5, $\alpha(G[0, m'-1]) = s + 1$ and so $\chi(G[0, 2m'-1]) \ge \chi(G[0, m'-1]) \ge m'/(s+1)$. We only need to prove the case where $(s+1) \mid m'$. Assume to the contrary that $\chi(G[0, 2m'-1]) = m'/(s+1)$. Let c be an $m'/(s+1)$-coloring of $G[0, 2m'-1]$. For all $j \in [0, m'-1]$, denote by H_j the subgraph of $G[0, 2m'-1]$ induced by $[j, j+m'-1]$. So $\alpha(H_j) = s+1$ and each of the $m'/(s+1)$ colors of c is used at most, and thus exactly $s+1$ times in each H_j. It follows that $c(j) = c(j+m')$, for all $j \in [0, m'-1]$.

Let $C = \{z_1, z_2, \ldots, z_{s+1}\}$ be a color class of c restricted to $[0, m'-1]$, where $z_1 < z_2 \cdots < z_{s+1}$. By Lemma 2.9 or Lemma 2.10, if $z \in C$, then either $z - z_1$ or $z - (m' + z_1)$ is a multiple of k. Since $d = \gcd(k, m+1) = \gcd(k, m')$, we have $z \equiv z_1 \pmod d$. So for each color class $C = \{z_1, z_2, \ldots, z_{s+1}\}$ of c restricted to $[0, m'-1]$, there exist integers $\lambda_1, \lambda_2, \ldots, \lambda_s$ such that $C = \{z_1, z_1 + \lambda_1 d, z_1 + \lambda_2 d, \ldots, z_1 + \lambda_s d\}$.

Let $u = m'/d$. Partition the vertex set of $G[0, m'-1]$ into d subsets $V_i = \{j : j \in [0, m'-1] \text{ and } j \equiv i \pmod{d}\}, i \in [0, d-1]$, each of size u. Then each color class C of c restricted to $[0, m'-1]$ must be contained in some V_i. Consequently each of these d subsets V_i is the union of some color classes, each of which is of order $s+1$. That means $(s+1) \mid u$, or $d(s+1) \mid m'$. This contradiction shows that $\chi(G[0, 2m'-1]) > m'/(s+1)$. Recall that $D_{m,[k,sk+\beta]} \subseteq D_{m,k,s}$, by Theorem 1.3, $\chi(Z, D_{m,[k,sk+\beta]}) \leq \chi(Z, D_{m,k,s})$. Therefore $\chi(Z, D_{m,[k,sk+\beta]}) = m'/(s+1) + 1$. $\square$

Corollary 2.12. *Suppose $s \geq 2$ and $m \geq (s+2)k + \beta - 1$. Then*

$$\chi(Z, D_{m,[k,sk+\beta]}) = \begin{cases} \lceil m'/(s+1) \rceil, & \text{if } d = 1 \text{ or } d(s+1) \mid m'; \\ \lceil (m'+1)/(s+1) \rceil, & \text{if } d \neq 1 \text{ and } d(s+1) \nmid m'. \end{cases}$$

Theorem 2.13. *Suppose $s \geq 2, m \geq (s+2)k + \beta - 1$, $d \neq 1$ and $d(s+1) \nmid m'$. Let $\tau = \gcd(s+1, 2m'+1)$. Then*

$$\chi_c(Z, D_{m,[k,sk+\beta]}) \geq \begin{cases} (m'+1/3)/(s+1), & \\ (m'+1/2)/(s+1), & \text{if } \tau \neq 1; \\ m'/(s+1) + 1/(2s+1), & \text{if } \tau = 1 \text{ and } (s+1) \mid m'. \end{cases}$$

Proof. We consider the induced subgraph $G[0, 2m'-1]$. Suppose $\chi_c(G[0, 2m'-1]) = p/q$, where p and q are relatively prime. By Theorem 1.3 and the fact that $G(Z, D_{m,[k,sk+\beta]})$ is a sub-graph of $G(Z, D_{m,k,s})$, we have

$$\begin{aligned} &(m'+1)/(s+1) \geq \chi_c(Z, D_{m,[k,sk+\beta]}) \geq \chi_c(G[0, 2m'-1]) \\ &\quad = p/q \geq m'/(s+1). \end{aligned} \tag{16}$$

If $(s+1) \mid m'$ then, by Theorem 2.11, $p/q = \chi_c(G[0, 2m'-1]) > m'/(s+1)$ and p/q cannot be an integer. If $(s+1) \nmid m'$, then $p/q = \chi_c(G[0, 2m'-1])$ is an integer only if $\chi_c(Z, D_{m,[k,sk+\beta]}) = (m'+1)/(s+1)$ and is an integer. So we may assume that $q \geq 2$.

Let c be a (p, q)-coloring of $G[0, 2m'-1]$. Then c uses every one of the p colors [18]. For each $j \in [0, p-1]$, put $X_j = c^{-1}(j)$ and $Y_j = X_j \cup X_{j+1} \cup \cdots \cup X_{j+q-1}$. Clearly Y_j is an independent set of G. If $S \subset [0, 2m'-1]$ and $a < b$, then $S \cap [a, b]$ is written as $S[a, b]$.

Two elements $x_1, x_2 \in [0, 2m'-1]$ are *equivalent* if $x_1 \equiv x_2 \pmod{d}$, otherwise *non-equivalent*. We shall show that there exist $[i', i'+q-2] \subseteq [0, p-1]$ such that Y_j contains two non-equivalent elements if $j \in [i', i'+q-2]$. If X_i contains two non-equivalent elements, then $Y_j = X_j \cup X_{j+1} \cup \cdots \cup X_{j+q-1}$ contains the same pair for $j \in [i-q+1, i]$. If all elements in a single X_i are equivalent, we shall consider X_{i+1}, and subsequent sets if necessary, until we get two non-equivalent elements x_1 and x_2, $x_1 \in X_{q-1}$ and $x_2 \in X_q$ respectively (this is possible since $d \neq 1$). Then each of the $q-1$ sets in $\{Y_j \mid j \in [1, q-1]\}$ contains the same pair $\{x_1, x_2\}$, where $x_1 < x_2$. We may assume that $x_1 \leq m'-1$, otherwise it can be treated symmetrically. We consider two cases for any one of such Y_j:

Case 1: $x_2 - x_1 \leq m' - 1$.

If $x_1 + m' \in Y_j$, then $Y_j[x_1 + sk + 1, x_1 + m] \cup [x_1 + m' - k + 1, x_1 + m' - 1] = \varnothing$, and consequently $\mathcal{C} = Y_j[x_1, x_1 + m' - 1] \subseteq [x_1, x_1 + sk] \cup [x_1 + m + 1, x_1 + m' - k]$.

By Lemmas 2.9 and 2.10, all elements in $\mathcal{C}$ must be equivalent to each other, contradicting the fact that $x_2 \in Y_j[x_1, x_1 + m' - 1]$ and is not equivalent to x_1. Thus $x_1 + m' \notin Y_j$. Since $|Y_j[x_1, x_1 + m' - 1]| \leq s + 1$, therefore $|Y_j[x_1 + 1, x_1 + m']| \leq s$ and we have

$$p(s+1) - (q-1) \geq \sum_{j=0}^{p-1} |Y_j[x_1 + 1, x_1 + m']| = qm'. \tag{17}$$

Case 2: $x_2 - x_1 \geq m' + 1$.

If $x_2 - m' \in Y_j$, then Y_j contains the pair of non-equivalent elements x_1 and $x_2 - m'$, and following the argument in Case 1, we have $|Y_j[x_1 + 1, x_1 + m']| \leq s$. If $x_2 - m' \notin Y_j$, then since $|Y_j[x_2 - m' + 1, x_2]| \leq s + 1$, we have $|Y_j[x_2 - m', x_2 - 1]| \leq s$. Therefore one of the following must hold:

$$p(s+1) - \lceil (q-1)/2 \rceil \geq \sum_{j=0}^{p-1} |Y_j[x_1 + 1, x_1 + m']| = qm'. \tag{18}$$

$$p(s+1) - \lceil (q-1)/2 \rceil \geq \sum_{j=0}^{p-1} |Y_j[x_2 - m', x_2 - 1]| = qm'. \tag{19}$$

If q is even, then $p(s+1) - q/2 \geq qm'$, i.e.,

$$p/q \geq \frac{m' + (1/2)}{s+1}. \tag{20}$$

If q is odd, then $p(s+1) - (q-1)/2 \geq qm'$, i.e.,

$$p/q \geq \frac{m' + (1/2)}{s+1} - \frac{1}{2q(s+1)}. \tag{21}$$

Since the least odd value of q is 3, therefore

$$\chi_c(Z, D_{m,[k,sk+\beta]}) \geq \frac{m' + (1/3)}{s+1}. \tag{22}$$

Suppose $\tau \neq 1$. If q is odd, we reformulate (21) to

$$2p(s+1) \geq (2m' + 1)q - 1. \tag{23}$$

Since both $2p(s+1)$ and $(2m'+1)q$ are multiples of τ, (23) may also be improved to (20). Therefore

$$\chi_c(Z, D_{m,[k,sk+\beta]}) \geq \frac{m' + (1/2)}{s+1}. \tag{24}$$

Since $q \geq 2$, any one of (17), (18) or (19) implies that

$$p(s+1) > qm'. \tag{25}$$

If $(s+1) \mid m'$, then both $p(s+1)$ and qm' are multiples of $s+1$ and consequently (25) may be improved to

$$p(s+1) \geq qm' + (s+1). \tag{26}$$

Since $p \leq 2m'$, it follows that $q \leq 2s+1$ and (26) becomes

$$p/q \geq \frac{m'}{s+1} + \frac{1}{q} \geq \frac{m'}{s+1} + \frac{1}{(2s+1)}. \tag{27}$$

□

If $\beta = 0$, Theorem 2.13 may be improved to:

Theorem 2.14. *Suppose $s \geq 2$ and $m \geq (s+2)k - 1$. If $(s+1) \mid m'$ but $d(s+1) \nmid m'$, then $\chi_c(Z, D_{m,[k,sk]}) = (m'+1)/(s+1)$.*

Proof. Suppose $\chi_c(G[0, m'+k-1]) = p/q$ and c is a (p, q)-coloring of $G[0, m'+k-1]$. Because $\beta = 0$, we can modify the proof of Theorem 2.11 to get $\chi_c(G[0, m'+k-1]) > m'/(s+1)$. Similar to the proof of Theorem 2.13, we also have

$$\begin{aligned}(m'+1)/(s+1) &\geq \chi_c(Z, D_{m,[k,sk]}) \geq \chi_c(G[0, m'+k-1]) \\ &= p/q > m'/(s+1)\end{aligned} \tag{28}$$

and

$$p(s+1) > qm'. \tag{29}$$

Because $(s+1) \mid m'$, both sides of (29) are multiples of $(s+1)$, and hence

$$p(s+1) \geq qm' + (s+1). \tag{30}$$

Since $p \leq m' + k$ and $p/q > m'/(s+1)$, we have $s+1 \geq q$ and hence $p/q \geq (m'+1)/(s+1)$. □

3. $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$

In this section we investigate $\chi_f(Z, D_{m,[k,sk+\beta]})$ and $\chi_c(Z, D_{m,[k,sk+\beta]})$ when $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$. From now on, we denote $m - sk - \beta$ by m^*. Let δ and ε be two non-negative integers such that $sk + \beta = \delta(m^*+1) + \varepsilon$, where $0 \leq \varepsilon \leq m^*$. If A is a subset of Z, we denote the set $\{x + t : x \in D\}$ by $A + t$.

Lemma 3.1. *Suppose $s \geq 2$. If $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$, then*

$$\omega(G(Z, D_{m,[k,sk+\beta]})) = m^* + 1.$$

Proof. We can verify that $[0, k-1] \cup [(s+1)k + \beta, m]$ is a clique of $G(Z, D_{m,[k,sk+\beta]})$ with $m^* + 1$ vertices. So $\omega(G(Z, D_{m,[k,sk+\beta]})) \geq m^* + 1$, and we only need to prove the opposite inequality. Let A be a maximum clique. Without loss of generality, we assume $0 \in A$ and $A \cap Z^-$ is empty. It follows that $A \subseteq [0, k-1] \cup [sk + \beta + 1, m]$. For each $i \in [1, k-1]$, i is not adjacent to $i + sk + \beta$, therefore either $i \notin A$ or $i + sk + \beta \notin A$. So $|A \cap ([1, k-1] \cup [sk + \beta + 1, (s+1)k + \beta - 1])| \leq k - 1$, and the lemma follows. □

Lemma 3.2. *Let $G(Z, D)$ be a distance graph and p be an integral upper bound of D. If $G[0, p]$ has a maximum independent set A such that $A \cup A + (p+1)$ is an independent set of $G(Z, D)$, then $\chi_f(G(Z, D)) = (p+1)/|A|$.*

Proof. Clearly $\chi_f(G(Z, D)) \geq (p+1)/|A|$. To prove the opposite inequality, we give a fractional coloring of $G(Z, D)$ with weight $(p+1)/|A|$. For $j \in [0, p]$, let $I_j = \{i : |j - i| \pmod{p+1} \in A\}$. Since $p \geq \max\{x : x \in D\}$ and $A \cup A + (p+1)$ is independent, I_j is an independent set of $G(Z, D)$ for all $j \in [0, p]$. We can check that every integer belongs to exactly $|A|$ of the independent sets I_j, $0 \leq j \leq p$. For $I \in \Gamma(G(Z, D))$, we define a mapping c from $\Gamma(G(Z, D))$ to $[0, 1]$ as follows:

$$c(I) = \begin{cases} 1/|A| & \text{if } I = I_j,\ j \in [0, p]; \\ 0 & \text{otherwise.} \end{cases}$$

Then c is a fractional coloring of $G(Z, D)$. This implies that $\chi_f(G(Z, D)) \leq (p+1)/|A|$. □

Theorem 3.3. *Suppose $s \geq 2$. If $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$ and $\varepsilon = 0$ then*

$$\chi_f(Z, D_{m,[k,sk+\beta]}) = \chi_c(Z, D_{m,[k,sk+\beta]}) = \chi(Z, D_{m,[k,sk+\beta]}) = m^* + 1.$$

Proof. For all $i \in [0, m^*]$, let $I_i = \{j \in Z : j \equiv i \mod (m^* + 1)\}$. We can check that each I_i is an independent set of $G(Z, D_{m,[k,sk+\beta]})$ and $\bigcup_{i=0}^{m-sk-\beta} I_i = Z$. It follows that $\chi(Z, D_{m,[k,sk+\beta]}) \leq m^* + 1$. The theorem follows from (1) and Lemma 3.1. □

Theorem 3.4. *Suppose $s \geq 2$ and $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$. If $\varepsilon \neq 0$ then*

$$m^* + 1 \leq \chi_f(Z, D_{m,[k,sk+\beta]}) \leq \frac{m+1}{\delta+1}.$$

Proof. Let $I = \{x(m^*+1) \mid x \in [0, \delta]\} \cup \{(m+1) + y(m^*+1) \mid y \in [0, \delta]\}$. We can see that I and $I \cup I^{+(2m+2)}$ are independent sets. From the proof of Lemma 3.2, we deduce that $\chi_f(Z, D_{m,[k,sk+\beta]}) \leq (m+1)/(\delta+1)$. The theorem follows from (1) and Lemma 3.1 and (1). □

Although we do not obtain the exact value of $\chi_f(Z, D_{m,[k,sk+\beta]})$ when $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$ and $\varepsilon \neq 0$, the upper and lower bounds given in Theorem 3.4 are sharp in some sense. The upper bound

$$\frac{m+1}{\delta+1} = \frac{m+1}{m+1-\varepsilon}(m^*+1) \to (m^*+1),$$

for every fixed k, as $m \to \infty$. Suppose k is fixed. If s, and consequently m, goes to infinity then $\chi_f(Z, D_{m,[k,sk+\beta]})$ is asymptotically equivalent to $m^* + 1$.

Theorem 3.5. *Suppose $s \geq 2$ and $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$. If $\varepsilon \neq 0$ then*

$$m^* + 2 \leq \chi(Z, D_{m,[k,sk+\beta]}) \leq \left\lceil \frac{m+1}{\delta+1} \right\rceil.$$

Proof. We first prove the lower bound. According to Lemma 3.1, $\omega(G(Z, D_{m,[k,sk+\beta]})) = m^* + 1$. Suppose to the contrary that $\chi(Z, D_{m,[k,sk+\beta]}) \leq m^* + 1$. Let c be an $(m^* + 1)$-coloring of $G(Z, D_{m,[k,sk+\beta]})$. For any $u \in Z$, denote by Cl_u the vertex set $[u, u+k-1] \cup [u+(s+1)k+\beta, u+m]$, which is a maximum clique of $G(Z, D_{m,[k,sk+\beta]})$ and $|c(Cl_u)| = m^* + 1$. Now consider Cl_{u+1}. Since $Cl_{u+1} = (Cl_u \setminus \{u, u + (s + 1)k + \beta\}) \cup \{u+k, u+m+1\}$, we have $\{c(u), c(u+(s+1)k+\beta)\} = \{c(u+k), c(u+m+1)\}$. Note that $u + m + 1$ is adjacent to $u + (s + 1)k + \beta$, it follows that $c(u) = c(u + m + 1)$ and $c(u + k) = c(u + (s + 1)k + \beta)$. As u is an arbitrary vertex of Z, we actually have $c(u) = c(u+sk+\beta) = c(u+m+1)$. Thus $c([u+k, u+m^*]) = c([u+(s+1)k+\beta, u+m])$ and $|c([u, u + m^*])| = m^* + 1$. This implies that $c(u) = c(u + m^* + 1)$ for any $u \in Z$. Therefore $c(u + sk + \beta) = c(u + \delta(m^* + 1) + \varepsilon) = c(u + \varepsilon)$. Since $c(u) = c(u + sk + \beta)$, we have $c(u) = c(u + \varepsilon)$. This contradicts the fact that $|c([u, u + m^*])| = m^* + 1$ because $1 \leq \varepsilon \leq m^*$.

To obtain the upper bound, we let λ and g be two integers such that $m+1 = \lambda(\delta+1)+g$, where $0 \leq g \leq \delta$. Then $\lfloor (m + 1)/(\delta + 1) \rfloor = \lambda$ and $\lceil (m + 1)/(\delta + 1) \rceil = \lambda + 1$. Let

$$I = \{0, \lambda + 1, 2(\lambda + 1), \ldots, g(\lambda + 1), (g + 1)\lambda + g, (g + 2)\lambda + g, \ldots, \delta\lambda + g\}$$

and partition Z into sets:

$$I_j = \{z \in Z\colon z \pmod{m + 1} \in I^{+j}\}, \qquad j \in [0, \lambda - 1] \qquad \text{and}$$
$$I_\lambda = \{z \in Z\colon z \pmod{m + 1} \in \{\lambda, 2\lambda + 1, \ldots, g\lambda + g - 1\}\}.$$

We can verify that I_j is an independent set of $G(Z, D_{m,[k,sk+\beta]})$ for all $j \in [0, \lambda]$. This gives the upper bound of $\chi(Z, D_{m,[k,sk+\beta]})$. $\square$

Note that if $\delta \geq \varepsilon - 1$ then $(m + 1)/(\delta + 1) \leq (m^* + 2)$. Hence by Theorem 3.5, we have

Corollary 3.6. *Suppose $s \geq 2$ and $(s + 1)k + \beta \leq m \leq (s + 2)k + \beta - 2$. If $\delta \geq \varepsilon - 1$, then*

$$\chi(Z, D_{m,[k,sk+\beta]}) = (m^* + 2).$$

Note that $0 \leq \varepsilon \leq m^* \leq 2k - 2$. If $k = 2$ then $0 \leq \varepsilon \leq 2$. Since $\delta \geq 1$, we have the following:

Corollary 3.7. *Suppose $s \geq 2$ and $(s + 1)k + \beta \leq m \leq (s + 2)k + \beta - 2$. If $k = 2$ then*

$$\chi(Z, D_{m,[k,sk+\beta]}) = (m^* + 2).$$

We have completely determined $\chi(G(Z, D_{m,[2,k']}))$ for arbitrary integers m and k'. In general, except when $s \geq 2$, $(s + 1)k + \beta \leq m \leq (s + 2)k + \beta - 2$ and $1 \leq \delta \leq \varepsilon - 2$, the exact values of $\chi(G(Z, D_{m,[k,sk+\beta]}))$ have also been determined.

We believe that $\chi_f(Z, D_{m,[k,sk+\beta]})$ is more likely to be the upper bound $(m+1)/(\delta+1)$. If we could prove that, then we would have completely determined $\chi_f(G(Z, D_{m,[k,k']}))$ and $\chi(G(Z, D_{m,[k,k']}))$ for arbitrary m, k, k' with $k' \geq k$.

4. Concluding remarks

The chromatic numbers of the two distance graphs $G_1 = G(Z, D_{m,k,s})$ and $G_2 = G(Z, D_{m,[k,sk+\beta]})$ are equal under many circumstances, but they may also differ at times.

Suppose $m < (s+1)k$. Then $\chi_f(G_1) = \chi_f(G_2) = k$ and $\chi_c(G_1) = \chi_c(G_2) = k$ by Theorems 1.1 and 2.1. Suppose $m \geq (s+2)k + \beta - 1$ and $s \geq 2$. Then $\chi_f(G_1) = \chi_f(G_2) = m'/(s+1)$ by Theorems 1.2 and 2.7; and $\chi(G_1) = \chi(G_2)$ by Theorem 1.3 and Corollary 2.12. If in addition, $d = 1$ or $d(s+1) \mid m'$, then $\chi_c(G_1) = \chi_c(G_2) = m'/(s+1)$ by Theorems 1.3 and 2.8. If $d \neq 1$ or $d(s+1) \nmid m'$, then $\chi_c(G_1) = (m'+1)/(s+1)$ by Theorem 1.3; but our results on $\chi_c(G_2)$ are not conclusive. We do have $\chi_c(G_2) = (m'+1)/(s+1)$, and thus $\chi_c(G_1) = \chi_c(G_2)$ if, in addition, $\beta = 0$ and $(s+1) \mid m'$—Theorem 2.14. Otherwise, we only have $\chi_c(G_2) \geq (m'+\epsilon)/(s+1)$, where $0 < \epsilon \leq 1/2$—Theorem 2.13. However, we incline to believe that in fact, ϵ should be equal to 1 and so $\chi_c(G_1) = \chi_c(G_2)$.

Suppose $s \geq 2$ and $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$. Then chromatic numbers of G_1 are in general not the same as G_2. Theorem 3.4 implies that $\chi_f(G_2)$ is asymptotically equal to m^*+1 as $m \to \infty$, and Theorem 1.2 says that $\chi_f(G_1) = m'/(s+1)$. To show that m^*+1 is strictly less than $m'/(s+1)$, we let $m = (s+1)k + \beta + r$, where $0 \leq r \leq k-2$. Then $m^*+1 = k+r+1 = 2k-(k-r-1)$, and

$$\frac{m'}{s+1} = \frac{(s+1)k+\beta+r+sk+1}{s+1} = 2k - \frac{k-r-\beta-1}{s+1}.$$

Since $k-r-1 > (k-r-\beta-1)/(s+1)$, we have $m'/(s+1) > m^*+1$. So if s, and hence m, is large enough, then $\chi_f(G_2) < \chi_f(G_1)$ provided $(s+1)k+\beta \leq m \leq (s+2)k+\beta-2$.

From Theorem 1.2 and (1), we also have $m'/(s+1) = \chi_f(G_1) \leq \chi(G_1)$. If $\delta \geq \varepsilon - 1$, then $\chi(G_2) = m^*+2$ by Corollary 3.6. Since $\lceil (m')/(s+1) \rceil = 2k - \lfloor (k-r-\beta-1)/(s+1) \rfloor$, $m^*+2 = 2k-(k-r-2)$, and $k-r-2 > \lfloor (k-r-\beta-1)/(s+1) \rfloor$ provided $r < k-2$ or $\beta \geq 2$, we have $m^*+2 < \lfloor m'/(s+1) \rfloor$. Therefore $\chi(G_2) < \chi_f(G_1) \leq \chi_c(G_1) \leq \chi(G_1)$.

If $(s+1)k \leq m \leq (s+1)k+\beta-1$ and $\beta \neq 0$, then $\chi_f(G_2) = \chi_c(G_2) = (m+1)/(s+1)$ by Theorem 2.3. It follows that $\chi_f(G_2) < \chi_f(G_1)$, $\chi_c(G_2) < \chi_c(G_1)$, and $\chi(G_2) < \chi(G_1)$.

We conclude this paper by asking the following two questions:

Question 1. For $(s+1)k + \beta \leq m \leq (s+2)k + \beta - 2$, what are $\chi_f(G(Z, D_{m,[k,sk+\beta]}))$, $\chi(G(Z, D_{m,[k,sk+\beta]}))$, and $\chi_c(G(Z, D_{m,[k,sk+\beta]}))$?

Question 2. Suppose $m \geq (s+2)k + \beta - 1$, $d = \gcd(m', k) \neq 1$, and $d(s+1) \nmid m'$. Is it true that $\chi_c(G(Z, D_{m,[k,sk+\beta]})) = (m'+1)/(s+1)$?

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